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# Majorization Relations for Hadamard Products*

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Dedicated to Professor M. Fiedler and Professor V. Pták with admiration.

Submitted by Wayne Barrett

ABSTRACT

We settle affirmatively the conjecture of Johnson and Bapat on the Hadamard product $A \circ B$ of positive definite matrices $A$, $B$:

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(AB) \qquad (k = 1, 2, \dots, n).$$

## 1. INTRODUCTION AND THEOREM

Given an $n \times n$ Hermitian matrix $A$, let us always arrange its eigenvalues $\lambda_i(A)$ $(i = 1, 2, \dots, n)$ in decreasing order

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

*Another proof of the Bapat/Johnson conjecture appears in G. Visick's "A Weak Majorization Involving the Matrices $A \cdot B$ and $AB$," on pages 731–744 of this issue.

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Let us write $A \succ B$ for Hermitian $A$, $B$ to mean the majorization $[\lambda_i(A)] \succ [\lambda_i(B)]$, that is,

$$\sum_{i=1}^{k} \lambda_i(A) \geq \sum_{i=1}^{k} \lambda_i(B) \qquad (k = 1, 2, \ldots, n)$$

and

$$\sum_{i=1}^{n} \lambda_i(A) = \sum_{i=1}^{n} \lambda_i(B).$$

Remark that

$$A \succ B \quad \Rightarrow \quad \sum_{i=k}^{n} \lambda_i(A) \leq \sum_{i=k}^{n} \lambda_i(B) \qquad (k = 1, 2, \ldots, n). \tag{1}$$

When $A, B > 0$ (positive definite), let us write $A \underset{(\log)}{\succ} B$ to mean $\log A \succ \log B$, or equivalently

$$\prod_{i=1}^{k} \lambda_i(A) \geq \prod_{i=1}^{k} \lambda_i(B) \qquad (k = 1, 2, \ldots, n)$$

and

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B).$$

Then it follows from the above remark that

$$A \underset{(\log)}{\succ} B \quad \Rightarrow \quad \prod_{i=k}^{n} \lambda_i(A) \leq \prod_{i=k}^{n} \lambda_i(B) \qquad (k = 1, 2, \ldots, n). \tag{2}$$

Given $A$, $B$ let us denote by $A \circ B$ their Hadamard (i.e., entrywise) product. It is obvious that $A \circ B = B \circ A$ and that when $I$ is the identity matrix, $A \circ I$ is the diagonal matrix produced by $A$.

The celebrated Schur theorem (see [10, p. 258]) says that

$$A, B \geq 0 \quad \Rightarrow \quad A \circ B \geq 0. \tag{3}$$

The main purpose of this paper is to prove the following majorization relation between the eigenvalues of $A \circ B$ and $AB$. Here remark that since $A^{1/2}BA^{1/2}$ is similar to $AB$, we can write $\lambda_i(A^{1/2}BA^{1/2})$ instead of $\lambda_i(AB)$.

THEOREM 1. *Let* $A, B > 0$. *Then*

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(AB) \qquad (k = 1, 2, \dots, n).$$

This theorem settles affirmatively the conjecture posed by Johnson and Bapat [8] (see also [11] and [7, p. 103]). Also, this theorem gives an improvement of the result of Bapat and Sunder [3]

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B) \qquad (k = 1, 2, \dots, n),$$

because according to the Horn theorem (see [10, p. 246])

$$\prod_{i=k}^{n} \lambda_i(AB) \geq \prod_{i=k}^{n} \lambda_i(A)\lambda_i(B) \qquad (k = 1, 2, \dots, n).$$

Using the method of the proof of Theorem 1, we can present the following variant.

THEOREM 2. *Let* $A, B > 0$. *Then*

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(AB^T) \qquad (k = 1, 2, \dots, n).$$

From Theorems 1 and 2 we can immediately derive the classical theorems of Fiedler [4, 5] that for $A, B > 0$

$$A \circ B \geq \lambda_n(AB)I \quad \text{and} \quad A \circ B \geq \lambda_n(AB^T)I. \tag{4}$$

In [1] we introduced the notion of *geometric mean* $A\#B$ for $A, B > 0$ as

$$A\#B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}.$$

It is shown in [2] (see also [6]) that $A^{1/2}BA^{1/2} \underset{(\log)}{\succ} (A\#B)^2$, so that we have by (2)

$$\prod_{i=k}^{n} \lambda_i(A\#B)^2 \geq \prod_{i=k}^{n} \lambda_i(AB) \qquad (k = 1, 2, \dots, n).$$

Therefore the following is an improvement of Theorem 1.

THEOREM 3. *Let* $A$, $B > 0$. *Then*

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(A \# B)^2 \qquad (k = 1, 2, \dots, n).$$

## 2. PROOFS

For the sake of completeness let us repeat some discussions in [1]. First, denote by $\mathbb{M}_n$ the space of all $n \times n$ complex matrices equipped with the cone $\mathbb{P}_n$ of positive semidefinite matrices. A linear map $\Phi$ from $\mathbb{M}_m$ to $\mathbb{M}_n$ is called *positive* if $\Phi(\mathbb{P}_m) \subset \mathbb{P}_n$. It is called *unital* if $\Phi(I_m) = I_n$, where $I_n$ is the identity matrix of order $n$. The following is a well-known fact from the theory of operator algebras (see [9, p. 770]).

LEMMA 4. *If $\Phi$ is a unital positive linear map, then*

$$\Phi(X^2) \geq \Phi(X)^2 \qquad (X > 0).$$

Since for $0 < \alpha < 1$ the nonlinear map on $\mathbb{P}_n : X \mapsto X^\alpha$ is order preserving (see [9, p. 464] and [7, p. 132]), we can derive from Lemma 4 that for $X > 0$

$$\Phi(X)^\alpha \geq \Phi(X^\alpha) \qquad (\alpha = 1/2^k, \quad k = 1, 2, \dots).$$

Since

$$\left. \frac{d}{d\alpha} X^\alpha \right|_{\alpha = 0} = \log X \qquad (X > 0),$$

this implies the following.

LEMMA 5. *If $\Phi$ is a unital positive linear map and $X > 0$, then*

$$\log \Phi(X) \geq \Phi(\log X).$$

*Proof of Theorem 1.* We can identify the tensor product $\mathbb{M}_n \otimes \mathbb{M}_n$ with $\mathbb{M}_{n^2}$. Then there is a unital positive linear map $\Phi_n$ such that

$$\Phi_n(A \otimes B) = A \circ B \qquad (A, B \in \mathbb{M}_n).$$

Since for $A, B > 0$

$$\begin{aligned}\log(A \otimes B) &= \frac{d}{d\alpha}(A \otimes B)^{\alpha}\bigg|_{\alpha=0} = \frac{d}{d\alpha}(A^{\alpha} \otimes B^{\alpha})\bigg|_{\alpha=0}\\ &= (\log A) \otimes I_n + I_n \otimes (\log B),\end{aligned}$$

we can derive from Lemma 5 that

$$\begin{aligned}\log(A \circ B) &\geq (\log A) \circ I_n + I_n \circ (\log B)\\ &= \{\log A + \log B\} \circ I_n,\end{aligned}$$

which implies

$$\begin{aligned}\log\left(\prod_{i=k}^{n} \lambda_i(A \circ B)\right) &= \sum_{i=k}^{n} \lambda_i(\log(A \circ B))\\ &\geq \sum_{i=k}^{n} \lambda_i(\{\log A + \log B\} \circ I_n) \qquad (k = 1, 2, \dots, n).\end{aligned} \tag{5}$$

According to Schur's theorem on majorization of the diagonal entries of a Hermitian matrix by its eigenvalues (see [10, p. 218]),

$$\log A + \log B \succ \{\log A + \log B\} \circ I_n,$$

so that by (1)

$$\sum_{i=k}^{n} \lambda_i(\{\log A + \log B\} \circ I_n) \geq \sum_{i=k}^{n} \lambda_i(\log A + \log B) \qquad (k = 1, 2, \dots, n). \tag{6}$$

On the other hand, as shown in [2] (see also [6]), we have

$$\log(A^{1/2}BA^{1/2}) \succ \log A + \log B,$$

so that by (1)

$$\sum_{i=k}^{n} \lambda_i(\log A + \log B) \geq \sum_{i=k}^{n} \lambda_i\big(\log(A^{1/2}BA^{1/2})\big)$$

$$= \log\left(\prod_{i=k}^{n} \lambda_i(AB)\right) \qquad (k = 1, 2, \ldots, n). \quad (7)$$

When combined with (5) and (6), equation (7) yields

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(AB) \qquad (k = 1, 2, \ldots, n),$$

which establishes Theorem 1. ∎

*Proof of Theorem 2.* Since

$$\log B^T = (\log B)^T,$$

we have

$$(\log B^T) \circ I = (\log B)^T \circ I = (\log B) \circ I.$$

Therefore in the above proof we can replace $\{\log A + \log B\} \circ I$ by $\{\log A + \log B^T\} \circ I$. This completes the proof of Theorem 2. ∎

*Proof of Theorem 3.* Since direct computation shows that $A^{-1/2}(A\#B)B^{-1/2}$ is unitary, we have by (3.7.1) of [7]

$$\begin{bmatrix} A & A\#B \\ A\#B & B \end{bmatrix} \geq 0 \quad \text{and} \quad \begin{bmatrix} B & A\#B \\ A\#B & A \end{bmatrix} \geq 0.$$

Then by (3) we have

$$\begin{bmatrix} A \circ B & (A\#B)\circ(A\#B) \\ (A\#B)\circ(A\#B), & A \circ B \end{bmatrix} \geq 0,$$

which implies via (3.7.1) of [7] again

$$A \circ B \geq (A\#B)\circ(A\#B),$$

so that

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i((A\#B)\circ(A\#B)) \qquad (k = 1, 2, \ldots, n).$$

Finally, applying Theorem 1 to the right hand side, we have

$$\prod_{i=k}^{n} \lambda_i((A\#B)\circ(A\#B)) \geq \prod_{i=k}^{n} \lambda_i(A\#B)^2 \qquad (k = 1, 2, \ldots, n).$$

This completes the proof of Theorem 3. ∎

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